Galerkin and Reduced Basis methods for UQ

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Objectives of the lecture

- Basic principle of stochastic Galerkin projection
- Discuss derivation and elementary building blocks of the Galerkin projection
- Galerkin linear models and evaluation of non-linearities
- PGD and reduced basis methods.

Galerkin projection

- **Weak solution of the stochastic problem** $\mathcal{M}(U(\xi); D(\xi)) = 0$
- Needs **adaptation of deterministic codes**
- Potentially more efficient than NI techniques.
- Better suited to improvement (error estimate, optimal and basis reduction, . . .), thanks to functional analysis.

Stochastic discretization

Let $\mathcal{S}^{\mathrm{P}} \subset L^2(\Xi, p_\xi)$ defined as

$$\mathcal{S}^{\mathrm{P}} = \mathrm{span}\{\Psi_0, \ldots, \Psi_{\mathrm{P}}\},$$

where the $\{\Psi_k\}$ are orthogonal functionals in $\boldsymbol{\xi}$, *e.g.* a PC basis truncated to an order No.

$\mathcal{S}^{\mathrm{P}}$ is called the stochastic approximation space

We seek for the approximate stochastic model solution in $\mathcal{V} \otimes \mathcal{S}^{\mathrm{P}}$.

$$U(\boldsymbol{\xi}) \approx U^{\mathrm{P}}(\boldsymbol{\xi}) = \sum_{k=0}^{\mathrm{P}} u_k \Psi_k(\boldsymbol{\xi}).$$

Inserting U^{P} in the weak formulation yields the stochastic residual

$$\left\langle \mathcal{M}(U^{\mathrm{P}}(\boldsymbol{\xi}); D(\boldsymbol{\xi})), \beta(\boldsymbol{\xi}) \right\rangle = \left\langle R(U^{\mathrm{P}}), \beta \right\rangle.$$

Galerkin projection

[Ghanem & Spanos, 1991]

$$\left\langle \mathcal{M}(U^{\mathrm{P}}(\boldsymbol{\xi}); D(\boldsymbol{\xi})), \beta(\boldsymbol{\xi}) \right\rangle = \left\langle R(U^{\mathrm{P}}), \beta \right\rangle .$$

In general, one cannot find $U^{\mathrm{P}} \in \mathcal{V} \otimes \mathcal{S}^{\mathrm{P}}$ such that

$$\left\langle R(U^{\mathrm{P}}), \beta \right\rangle = 0 \quad \forall \beta \in L^2(\Xi, p_{\boldsymbol{\xi}}).$$

It is then required that $R(U^{\mathrm{P}})$ is orthogonal to the stochastic approximation space:

$$\left\langle \mathcal{M}(U^{\mathrm{P}}(\boldsymbol{\xi}); D(\boldsymbol{\xi})), \beta(\boldsymbol{\xi}) \right\rangle = 0 \quad \forall \beta \in \mathcal{S}^{\mathrm{P}}.$$

- This weak formulation corresponds to the stochastic Galerkin formulation.
- The actual formulation is obtained in practice by projecting all model equations on $\mathcal{S}^{\mathrm{P}}$ (see examples later).

The Galerkin projection results in a set of $\mathrm{P}+1$ coupled problems for the stochastic modes u_k of the solution.

Find $\{u_k, k=0,\dots,\mathrm{P}+1\} \in \mathcal{V}^{\mathrm{P}+1}$ such that

$$\left\langle \mathcal{M}\left(\sum_{k=0}^{\mathrm{P}} u_k \Psi_k(\xi); D(\xi)\right), \Psi_l(\xi)\right\rangle = 0, \quad l=0,\dots,\mathrm{P}.$$

- The size of the Galerkin problem increases with P.
- Recall that $\mathrm{P} = 1 = (\mathrm{N}+\mathrm{No})!/\mathrm{N}!\mathrm{No}!$ for polynomial truncation at order No.
- This can be very costly for complex problems requiring large parametrization and large expansion order.
- Projections on the Ψ_l of the model equations can be problematic in presence of non-linearities.

The Galerkin projection for the elliptic problem:
Find $U(\boldsymbol{x}, \xi) \in H_0^1 \otimes L^2(\Xi, P_\Xi)$ such that

$$A(U, V; D) = B(V) \quad \forall V(\boldsymbol{x}, \xi) \in H_0^1 \otimes L^2(\Xi, P_\Xi),$$

where

$$A(U, V; D) = \mathbb{E}\left[\int_\Omega \nu(\boldsymbol{x}, \xi) \nabla U(\boldsymbol{x}, \xi) \cdot \nabla V(\boldsymbol{x}, \xi) d\boldsymbol{x}\right], \quad B(V) = \mathbb{E}\left[\int_\Omega F(\boldsymbol{x}, \xi) V(\boldsymbol{x}, \xi) d\boldsymbol{x}\right].$$

Introducing the PC expansion of U, it comes the **coupled** set of deterministic problems:
Find $\{u_k\}_{k=0,\dots,\mathrm{P}} \in (H_0^1)^{\mathrm{P}+1}$ such that

$$\sum_{l=0}^{\mathrm{P}} a_{kl}(u_l, v) = b_k(v) \quad \forall v \in H_0^1, k = 0, \dots, \mathrm{P},$$

where

$$a_{kl}(u, v) = \int_\Omega \mathbb{E}\left[\nu(\boldsymbol{x}, \xi)\Psi_k(\xi)\Psi_l(\xi)\right] \nabla u \cdot \nabla v d\boldsymbol{x}, \quad b_k(v) = \int_\Omega \mathbb{E}\left[f(\boldsymbol{x}, \xi)\Psi_k(\xi)\right] v(\boldsymbol{x}) d\boldsymbol{x}.$$

Galerkin projection of discrete deterministic problems

The previous development can be applied to models discretized at the deterministic level.

Seeking for for $\boldsymbol{U}(\xi) \approx \boldsymbol{U}^{\mathrm{P}} \in \mathbb{R}^m \otimes \mathcal{S}^{\mathrm{P}}$, we obtain Find $\{\boldsymbol{u}_k, k = 0, \ldots, \mathrm{P}+1\} \in (\mathbb{R}^m)^{\mathrm{P}+1}$ such that

$$\left\langle \mathcal{M}_h \left(\sum_{k=0}^{\mathrm{P}} \boldsymbol{u}_k \Psi_k(\xi); D(\xi) \right), \Psi_l(\xi) \right\rangle = 0, \quad l = 0, \ldots, \mathrm{P}.$$

For many models, apply the stochastic discretization before the deterministic discretization results in the same Galerkin problem as proceeding the reverse way, provided that $\mathcal{V}^h$ is independent of ξ. Exceptions include, *e.g.*,

- Lagrangian formulations [OLM & OK, JCP 2009],
- treatment of geometric uncertainties.

- The linear Galerkin problem couples all the stochastic modes $\boldsymbol{u}_i \in \mathbb{R}^m$ of the stochastic solution.
- It is not possible in general to compute independently the components $\boldsymbol{u}_i$.
- The size of the spectral problem is large: $m \times \dim \mathcal{S}^{\mathrm{P}} = m \times (\mathrm{P}+1)$.
- Resolution of the linear Galerkin system can be demanding.
- An understanding of the block structured system is instructive to design and apply well-suited numerical methods.

Structure of Galerkin problems for uncertain linear operators

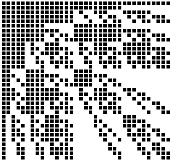
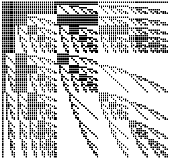
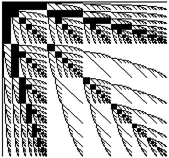
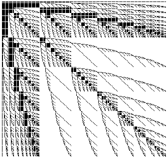
$\mathrm{N} = 4$-dim $\mathcal{S}^{\mathrm{P}} = 35$-$S = 0.58$	$\mathrm{N} = 6$-dim $\mathcal{S}^{\mathrm{P}} = 84$-$S = 0.41$
	
$\mathrm{N} = 8$-dim $\mathcal{S}^{\mathrm{P}} = 165$-$S = 0.31$	$\mathrm{N} = 10$-dim $\mathcal{S}^{\mathrm{P}} = 286$-$S = 0.23$
	

Illustration of the sparse structure of the matrices of the linear spectral problem for different dimensions, N, with $\mathrm{No} = 3$. Matrix blocks $[\bar{A}]_{ij}$ that are generally non-zero appear as black squares.

Structure of Galerkin problems for uncertain linear operators

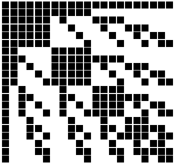
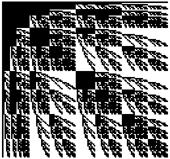
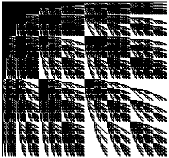
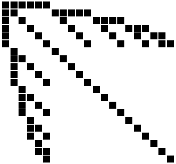
$\mathrm{No} = 2$-dim $\mathcal{S}^{\mathrm{P}} = 21$-$S = 0.52$	$\mathrm{No} = 3$-dim $\mathcal{S}^{\mathrm{P}} = 56$-$S = 0.49$
	
$\mathrm{No} = 4$-dim $\mathcal{S}^{\mathrm{P}} = 126$-$S = 0.54$	$\mathrm{No} = 5$-dim $\mathcal{S}^{\mathrm{P}} = 252$-$S = 0.55$
	

Illustration of the sparse structure of the matrices of the linear spectral problem for different expansion orders No, with $\mathrm{N} = 5$. Matrix blocks $[\bar{A}]_{ij}$ that are generally non-zero appear as black squares.

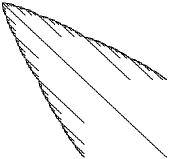
- Examples above assumes that $[A](\xi)$ has a full spectrum in $\mathcal{S}^{\mathrm{P}}$.
- When $[A](\xi)$ has a first-order expansion, the block structure of the linear spectral problem becomes even sparser.
- This behavior motivates the selection, whenever possible, of an approximation based on a first order operator.

Structure of Galerkin problems for uncertain linear operators

$\mathrm{No}=2$-$\dim \mathcal{S}^{\mathrm{P}}=21$-$S=0.184$ $\quad$ $\mathrm{No}=3$-$\dim \mathcal{S}^{\mathrm{P}}=56$-$S=0.084$



$\mathrm{No}=4$-$\dim \mathcal{S}^{\mathrm{P}}=126$-$S=0.043$ $\quad$ $\mathrm{No}=5$-$\dim \mathcal{S}^{\mathrm{P}}=252$-$S=0.024$



Case of a linear stochastic operator $[A](\boldsymbol{\xi})$ having a first-order expansion.

- The main difficulty in solving discrete linear spectral problems is the size of the system.
- The structure and sparsity of the linear Galerkin problem suggests iterative solution strategies.
- Iterative solvers (*e.g.* conjugate gradient techniques for symmetric systems, and Krylov subspace methods) can be used.
- The efficiency of iterative solvers depends on the availability of appropriate preconditioners which need be adapted to the Galerkin problem.
- Construction of the **preconditioners** to exploit the block-structure of the linear Galerkin problem.

Many models involve non-linearities of various types and their treatment is critical in stochastic Galerkin methods

Let $\{\Psi_k(\xi)\}_{k=0}^{\mathrm{P}}$ be an orthogonal basis of $\mathcal{S}^{\mathrm{P}} \subset L_2(\Xi, P_\Xi)$, and f a non-linear functional $u, v, \dots$:

$$u, v, \dots \in \mathbb{R} \mapsto f(u, v, \dots) \in \mathbb{R}.$$

For random arguments, $U(\xi), V(\xi), \dots \in \mathbb{R} \otimes \mathcal{S}^{\mathrm{P}}$, we generally have $f(U, V, \dots) =: G(\xi) \notin \mathbb{R} \otimes \mathcal{S}^{\mathrm{P}}$, but if $G(\xi) \in \mathbb{R} \otimes L_2(\Xi, P_\Xi)$ it has an orthogonal projection on $\mathcal{S}^{\mathrm{P}}$,

$$G(\xi) \approx \widehat{G} = \sum_{k=0}^{\mathrm{P}} g_k \Psi_k, \quad g_k = \frac{\langle f(U, V, \dots), \Psi_k \rangle}{\langle \Psi_k^2 \rangle}.$$

The problem is therefore to derive efficient strategies to compute the expansion coefficients g_k of $\hat{G}(\xi)$ from the expansion coefficients of its arguments $U(\xi), V(\xi), \dots$.

Polynomial non-linearities

The product of two quantities appears in many models.

It corresponds to the case $G(\xi) = W(\xi) = U(\xi)V(\xi)$ for $U, V \in \mathcal{S}^{\mathrm{P}}$ having known expansions. Clearly,

$$W(\xi) = \sum_{i=0}^{\mathrm{P}} \sum_{j=0}^{\mathrm{P}} u_i v_j \Psi_i(\xi)\Psi_j(\xi).$$

and in general $W(\xi) \notin \mathcal{S}^{\mathrm{P}}$ though it is in $L_2(\Xi, P_\Xi)$. Therefore, $\widehat{W}$, the orthogonal projection of W on $\mathcal{S}^{\mathrm{P}}$, has expansion coefficients

$$w_k = \frac{\langle W, \Psi_k \rangle}{\langle \Psi_k^2 \rangle} = \sum_{i=0}^{\mathrm{P}} \sum_{j=0}^{\mathrm{P}} u_i v_j C_{ijk}.$$

The result of the orthogonal projection of UV is called the Galerkin product of U and V and is denoted $U * V$.
The Galerkin product introduces **truncation errors** by disregarding the components of UV orthogonal to $\mathcal{S}^{\mathrm{P}}$.

Polynomial non-linearities

Higher order polynomial non-linearities are also frequent.
Consider first the triple product $G(\xi) = U(\xi)V(\xi)W(\xi)$ One can again perform an exact Galerkin projection of the triple product:

$$\widehat{UVW} := \sum_{m=0}^{\mathrm{P}} \widehat{uvw}_m \Psi_m = \sum_{m=0}^{\mathrm{P}} \Psi_m \left(\sum_{j,k,l=0}^{\mathrm{P}} T_{jklm} u_j v_k w_l \right),$$

$$T_{jklm} \equiv \frac{\langle \Psi_j \Psi_k \Psi_l \Psi_m \rangle}{\langle \Psi_m \Psi_m \rangle}.$$

- This exact Galerkin projection of the triple product involves the **fourth order tensor** T_{jklm}.
- T_{jklm} is sparse with many symmetries .
- However, computation and storage of T_{jklm} becomes quickly prohibitive when P increases.
- The exact Galerkin projection can hardly be extended further to higher order polynomial non-linearities.

Polynomial non-linearities

It is often preferred to rely on approximations for polynomial non-linearities of order larger than 2. For the triple product, an immediate approximation is

$$\widehat{UVW} \approx U * (V * W) = \widehat{U\widehat{VW}}.$$

This strategy can be extended to higher degree polynomial non-linearities by using successive Galerkin products. For instance,

$$\widehat{ABC \ldots D} \approx A * (B * (C * (\ldots * D))).$$

This procedure does not provide the exact Galerkin projection, since every intermediate product disregards the part orthogonal to $\mathcal{S}^{\mathrm{P}}$. Even for the triple product it is remarked that, in general

$$U * (V * W) \neq (U * V) * W \neq (U * W) * V.$$

The order in which the successive Galerkin products are applied affects the result.

Inverse and square root

Inverse and division are also common non-linearities.
For the inversion, one has to determine the expansion coefficients of the inverse U^{-1} of $U(\xi)$,

$$U^{-1}(\xi) = \frac{1}{U(\xi)} = \left(\sum_{k=0}^{\mathrm{P}} u_k \Psi_k(\xi) \right)^{-1},$$

such that

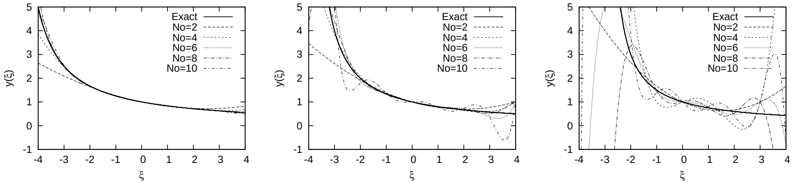
$$U^{-1}(\xi) U(\xi) = 1 \quad \text{a.s.}$$

U^{-1} is sought in $\mathcal{S}^{\mathrm{P}}$ and the previous equation needs to be interpreted in a weak sense. Using the Galerkin multiplication tensor, it comes

$$\begin{pmatrix} \sum_{j=0}^{\mathrm{P}} C_{j00} u_j & \dots & \sum_{j=0}^{\mathrm{P}} C_{j\mathrm{P}0} u_j \\ \vdots & \ddots & \vdots \\ \sum_{j=0}^{\mathrm{P}} C_{j0\mathrm{P}} u_j & \dots & \sum_{j=0}^{\mathrm{P}} C_{j\mathrm{P}\mathrm{P}} u_j \end{pmatrix} \begin{pmatrix} u_0^{-1} \\ \vdots \\ u_{\mathrm{P}}^{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}.$$

Due to truncature error, the above definition corresponds to the pseudo-spectral inverse U^{*-1} of U.

Inverse and square root



Pseudo-spectral approximation at different orders of the inverse $Y(\xi) = \widehat{U^{-1}}(\xi)$ of $U(\xi) = 1 + \alpha\xi$ with $\xi \sim \mathrm{N}(0, 1)$: $\alpha = 1/5$ (left), $1/4$ (center) and $1/3$ (right). Wiener-Hermite expansions are used.

Extend immediately to the evaluation of U/V

Inverse and square root

The Galerkin product can also serve to approximate square roots.
Given $U(\xi) > 0$ we have

$$U^{1/2}(\xi)U^{1/2}(\xi) = U(\xi).$$

The approximate $U^{*1/2} \in \mathcal{S}^{\mathrm{P}}$ of $U^{1/2}$ solves

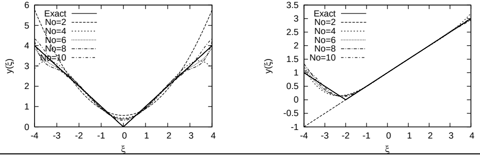
$$\begin{pmatrix} \sum_{j=0}^{\mathrm{P}} C_{j00} u^{1/2}{}_j & \cdots & \sum_{j=0}^{\mathrm{P}} C_{j\mathrm{P}0} u^{1/2}{}_j \\ \vdots & \ddots & \vdots \\ \sum_{j=0}^{\mathrm{P}} C_{j0\mathrm{P}} u^{1/2}{}_j & \cdots & \sum_{j=0}^{\mathrm{P}} C_{j\mathrm{P}\mathrm{P}} u^{1/2}{}_j \end{pmatrix} \begin{pmatrix} u^{1/2}{}_0 \\ \vdots \\ u^{1/2}{}_{\mathrm{P}} \end{pmatrix} = \begin{pmatrix} u_0 \\ \vdots \\ u_{\mathrm{P}} \end{pmatrix}.$$

This non-linear system can be solved using standard techniques (Newton-Raphson iterations) Choosing for the initial guess $U^{*1/2}(\xi) = \pm\sqrt{u_0}$ allows for the selection of the positive or negative square root of $U(\xi)$.

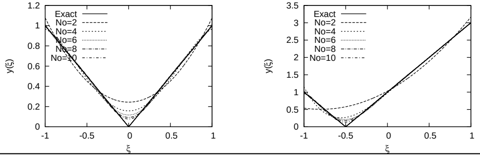
Absolute values

Application to the approximation of absolute values

$U(\xi) = \xi$	$U(\xi) = 1 + \xi/2$



$U(\xi) = \xi$	$U(\xi) = 1 + 2\xi$



Convergence with No of the pseudo-spectral approximation on $\mathcal{S}^{\mathrm{No}}$ of $Y(\xi) = |U(\xi)|$ for different $u(\xi)$. Top plots: $\xi \sim \mathrm{N}(0,1)$ and Wiener-Hermite expansions. Bottom plots: $\xi \sim \mathcal{U}(-1,1)$ and Wiener-Legendre expansions.

Other non-linearities

For sufficiently differentiable non-linearities one can rely on Taylor series

$$f(u) = f(\hat{u}) + (u - \hat{u})f'(\hat{u}) + \frac{(u-\hat{u})^2}{2} f''(\hat{u}) + \cdots$$

In the stochastic case, it is common to expand the series about the mean u_0 of U, at which $f'(u_0)$, $f''(u_0)$, $\cdots$ can be evaluated.
Successive powers of $\delta U := U - u_0$ can be evaluated in a pseudo-spectral fashion

$$\mathcal{S} \ni F(U) \approx f(u_0) + \delta U f'(u_0) + \frac{\delta U * \delta U}{2} f''(u_0) + \frac{\delta U * \delta U * \delta U}{6} f'''(u_0) + \cdots$$

- Convergence of the approximation needs be carefully analyzed.
- Impact of the pseudo spectral error is critical.
- Radius of convergence often unknown.

Other non-linearities

Integration approach for differentiable non-linearities [Debusschere *et al*, 2004]

If $f(\cdot)$ is analytical with derivative $f'(\cdot)$, f can be defined as some integral of f' along a deterministic integration path.

Let $Y(s,\xi)$ be a stochastic processes of $L^2(\Xi, P_\Xi)$, and consider $G(s,\xi) := f(Y)$:

$$Y = Y(s,\xi) = \sum_{k=0}^{\mathrm{P}} y_k(s)\Psi_k(\xi), \quad G = G(s,\xi) = \sum_{k=0}^{\mathrm{P}} g_k(s)\Psi_k(\xi).$$

Therefore, we have

$$\begin{aligned} \int_{s_1}^{s_2} \frac{\partial G}{\partial s} \mathrm{d}s &= \int_{s_1}^{s_2} G' \frac{\partial Y}{\partial s} \mathrm{d}s \\ \sum_{k=0}^{\mathrm{P}} \Psi_k \int_{s_1}^{s_2} \frac{\mathrm{d}g_k}{\mathrm{d}s} \mathrm{d}s &= \sum_{k=0}^{\mathrm{P}} \Psi_k \left[g_k(s_2) - g_k(s_1) \right] \\ &= \sum_{i=0}^{\mathrm{P}} \sum_{j=0}^{\mathrm{P}} \Psi_i \Psi_j \int_{s_1}^{s_2} g_i'(s) \frac{\mathrm{d}y_j}{\mathrm{d}s} \mathrm{d}s. \end{aligned}$$

Other non-linearities

The integration path is set such that for all $k = 0, \ldots, \mathrm{P}$

$$Y(s_1, \xi) = \hat{U}, \quad Y(s_2, \xi) = U, \tag{1}$$

we obtain

$$F(U(\xi))_k = F(\hat{U})_k + \sum_{i=0}^{\mathrm{P}} \sum_{j=0}^{\mathrm{P}} C_{ijk} \int_{\hat{u}_j}^{u_j} f_i' \, \mathrm{d}y_j, \quad \forall k = 0, \ldots, \mathrm{P}.$$

Provided that

- the PC expansion of $F(\hat{U})$ is known,
- the PC expansion of $F'(\cdot)$ is easily computed along the integration path,

the computation of $F(U)$ amounts to solve a set of coupled ODEs.

Other non-linearities

Example: exponential $f(u) = \exp(u)$.
We simply set $Y(s, \xi) = sU(\xi)$, $s_1 = 0$ and $s_2 = 1$.
Since $\exp(u)' = u$, we obtain the the set of coupled ODEs:

$$\frac{\mathrm{d}g_k}{\mathrm{d}s} = \sum_{i=0}^{\mathrm{P}} \sum_{j=0}^{\mathrm{P}} C_{ijk} u_i g_k, \quad k = 0, \dots, \mathrm{P},$$

to be integrated up to $s = 1$ from the initial condition

$$g_k(s = 0) = \langle \exp 0, \Psi_k \rangle = \delta_{k,0} \quad k = 0, \dots, \mathrm{P}.$$

- Standard techniques for ODEs can be used.
- Integration and stochastic truncation error control is critical.

Other non-linearities

Non-intrusive projections

- For general non-linearities $F(U, V, \dots)$ it is possible to proceed by non-intrusive projection techniques:

$$f_k := \frac{\langle F(U, V, \dots), \Psi_k \rangle}{\langle \Psi_k^2 \rangle}. \tag{2}$$

- Results in hybrid Galerkin / non-intrusive approaches when used in intermediate step of a Galerkin projection method (case of complex non-linear model).

$$\nabla \cdot (\nu(U) \nabla U) = g \text{ with BCs.} \tag{3}$$

- Interest can be questionable.

Table of content

Separated representation

The rank-m PGD approximation of U is [Nouy, 2007, 2008, 2010]

$$U(\boldsymbol{x},\theta) \approx U^m(\boldsymbol{x},\theta) = \sum_{\alpha=1}^{m<\mathrm{P}} u_\alpha(\boldsymbol{x})\lambda_\alpha(\theta), \quad \lambda_\alpha \in \mathcal{S}^{\mathrm{P}},\ u_\alpha \in \mathcal{V}.$$

Interpretation: U is approximated on

- the stochastic reduced basis $\{\lambda_1, \ldots, \lambda_m\}$ of $\mathcal{S}^{\mathrm{P}}$
- the deterministic reduced basis $\{u_1, \ldots, u_m\}$ of $\mathcal{V}$

none of which is selected *a priori*

The questions are then:

- how to **define** the (deterministic or stochastic) reduced basis ?
- how to **compute** the reduced basis and the m-terms PGD of U ?

Optimal L_2-spectral decomposition

POD, KL decomposition

$$U^m(\boldsymbol{x},\theta) = \sum_{\alpha=1}^{m} u_\alpha(\boldsymbol{x})\lambda_\alpha(\theta) \text{ minimizes } \mathbb{E}\left[\|U^m - U\|^2_{L^2(\Omega)}\right]$$

The modes u_α are the m **dominant eigenvectors** of the kernel $\mathbb{E}\left[U(\boldsymbol{x},\cdot)U(\boldsymbol{y},\cdot)\right]$:

$$\int_\Omega \mathbb{E}\left[U(\boldsymbol{x},\cdot)U(\boldsymbol{y},\cdot)\right] u_\alpha(\boldsymbol{y})d\boldsymbol{y} = \beta u_\alpha(\boldsymbol{x}), \quad \|u_\alpha\|_{L^2(\Omega)} = 1.$$

The modes are orthonormal:

$$\lambda_\alpha(\theta) = \int_\Omega U(\boldsymbol{x},\theta)u_\alpha(\boldsymbol{x})d\boldsymbol{x}$$

However $U(\boldsymbol{x},\theta)$, so $\mathbb{E}\left[u(\boldsymbol{x},\cdot)u(\boldsymbol{y},\cdot)\right]$ is **not known!**

- Solve the Galerkin problem in $\mathcal{V}^h \otimes \mathcal{S}^{P'<P}$ to construct $\{u_\alpha\}$, and then solve for the $\{\lambda_\alpha \in \mathcal{S}^P\}$.
- Solve the Galerkin problem in $\mathcal{V}^H \otimes \mathcal{S}^P$ to construct $\{\lambda_\alpha\}$, and then solve for the $\{u_\alpha \in \mathcal{V}^h\}$ with $\dim \mathcal{V}^H \ll \dim \mathcal{V}^h$.

See works by groups of Ghanem and Matthies.

Alternative definition of optimality

$A(\cdot,\cdot)$ is symmetric positive definite, so U minimizes the energy functional

$$\mathcal{J}(V) \equiv \frac{1}{2}A(V,V) - B(V)$$

We define U^m through

$$\mathcal{J}(U^m) = \min_{\{u_\alpha\},\{\lambda_\alpha\}} \mathcal{J}\left(\sum_{\alpha=1}^{m} u_\alpha \lambda_\alpha\right).$$

- Equivalent to minimizing a Rayleigh quotient
- Optimality w.r.t the A-norm (change of metric):

$$\|V\|_A^2 = \mathbb{E}\left[a(V,V)\right] = A(V,V)$$

Sequential construction:

For $i = 1, 2, 3 \ldots$

$$\mathcal{J}(\lambda_i u_i) = \min_{v \in \mathcal{V}, \beta \in \mathcal{S}^{\mathrm{P}}} \mathcal{J}\left(\beta v + \sum_{j=1}^{i-1} \lambda_j u_j\right) = \min_{v \in \mathcal{V}, \beta \in \mathcal{S}^{\mathrm{P}}} \mathcal{J}\left(\beta v + U^{i-1}\right)$$

The optimal couple (λ_i, u_i) solves simultaneously

- a) deterministic problem $\qquad u_i = \mathcal{D}(\lambda_i, U^{i-1})$

$$A(\lambda_i u_i, \lambda_i v) = B(\lambda_i v) - A\left(U^{i-1}, \lambda_i v\right), \quad \forall v \in \mathcal{V}$$

- b) stochastic problem $\qquad \lambda_i = \mathcal{S}(u_i, U^{i-1})$

$$A(\lambda_i u_i, \beta u_i) = B(\beta u_i) - A\left(U^{i-1}, \beta u_i\right), \quad \forall \beta \in \mathcal{S}^{\mathrm{P}}$$

Sequential construction:

For $i = 1, 2, 3 \dots$

$$\mathcal{J}(\lambda_i u_i) = \min_{v \in \mathcal{V}, \beta \in \mathcal{S}^{\mathrm{P}}} \mathcal{J}\left(\beta v + \sum_{j=1}^{i-1} \lambda_j u_j\right) = \min_{v \in \mathcal{V}, \beta \in \mathcal{S}^{\mathrm{P}}} \mathcal{J}\left(\beta v + U^{i-1}\right)$$

The optimal couple (λ_i, u_i) solves simultaneously

- a) deterministic problem $\qquad u_i = \mathcal{D}(\lambda_i, U^{i-1})$

$$\int_\Omega \mathbb{E}\left[\lambda_i^2 k\right] \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} v \mathrm{d}\boldsymbol{x} = \mathbb{E}\left[-\int_\Omega \lambda_i k \boldsymbol{\nabla} U^{i-1} \cdot \boldsymbol{\nabla} v \mathrm{d}\boldsymbol{x} + \int_\Omega \lambda_i f v \mathrm{d}\boldsymbol{x}\right], \quad \forall v.$$

- b) stochastic problem $\qquad \lambda_i = \mathcal{S}(u_i, U^{i-1})$

$$\mathbb{E}\left[\lambda_i \beta \int_\Omega k \boldsymbol{\nabla} u_i \cdot \boldsymbol{\nabla} u_i \mathrm{d}\boldsymbol{x}\right] = \mathbb{E}\left[-\beta\left(\int_\Omega k \boldsymbol{\nabla} U^{i-1} \cdot \boldsymbol{\nabla} u_i \mathrm{d}\boldsymbol{x} + \int_\Omega f u_i \mathrm{d}\boldsymbol{x}\right)\right], \quad \forall \beta.$$

Sequential construction:

For $i = 1, 2, 3 \dots$

$$\mathcal{J}(\lambda_i u_i) = \min_{v \in \mathcal{V}, \beta \in \mathcal{S}^{\mathrm{P}}} \mathcal{J}\left(\beta v + \sum_{j=1}^{i-1} \lambda_j u_j\right) = \min_{v \in \mathcal{V}, \beta \in \mathcal{S}^{\mathrm{P}}} \mathcal{J}\left(\beta v + U^{i-1}\right)$$

The optimal couple (λ_i, u_i) solves simultaneously

- a) deterministic problem $\quad u_i = \mathcal{D}(\lambda_i, U^{i-1})$

$$\int_\Omega \mathbb{E}\left[\lambda_i^2 k\right] \nabla u_i \cdot \nabla v \mathrm{d}\boldsymbol{x} = \mathbb{E}\left[-\int_\Omega \lambda_i k \nabla U^{i-1} \cdot \nabla v \mathrm{d}\boldsymbol{x} + \int_\Omega \lambda_i f v \mathrm{d}\boldsymbol{x}\right], \quad \forall v.$$

- b) stochastic problem $\quad \lambda_i = \mathcal{S}(u_i, U^{i-1})$

$$\mathbb{E}\left[\lambda_i \beta \int_\Omega k \nabla u_i \cdot \nabla u_i \mathrm{d}\boldsymbol{x}\right] = \mathbb{E}\left[-\beta\left(\int_\Omega k \nabla U^{i-1} \cdot \nabla u_i \mathrm{d}\boldsymbol{x} + \int_\Omega f u_i \mathrm{d}\boldsymbol{x}\right)\right], \quad \forall \beta.$$

- The couple (λ_i, u_i) is a fixed-point of:

$$\lambda_i = \mathcal{S} \circ \mathcal{D}(\lambda_i, \cdot), \quad u_i = \mathcal{D} \circ \mathcal{S}(u_i, \cdot)$$

$\Rightarrow$ arbitrary normalization of one of the two elements.

Algorithms inspired from dominant subspace methods
Power-type, Krylov/Arnoldi, ...

Power Iterations

1. Set $l = 1$
2. initialize λ (*e.g.* randomly)
3. While not converged, repeat (power iterations)
   - **a)** Solve: $u = \mathcal{D}(\lambda, U^{l-1})$
   - **b)** Normalize u
   - **c)** Solve: $\lambda = \mathcal{S}(u, U^{l-1})$
4. Set $u_l = u$, $\lambda_l = \lambda$
5. $l \leftarrow l + 1$, if $l < m$ repeat from step 2

Comments:

- Convergence criteria for the power iterations (subspace with $\dim > 1$ or clustered eigenvalues) [Nouy, 2007,2008]
- Usually few (4 to 5) inner iterations are sufficient

Power Iterations with Update

1. Same as Power Iterations, but after (u_l, λ_l) is obtained (step 4) update of the stochastic coefficients:
   - Orthonormalyze $\{u_1, \dots, u_l\}$ (optional)
   - Find $\{\lambda_1, \dots, \lambda_l\}$ s.t.

$$A\left(\sum_{i=1}^{l} u_i \lambda_i, \sum_{i=1}^{l} u_i \beta_i\right) = B\left(\sum_{i=1}^{l} u_i \beta_i\right), \quad \forall \beta_{i=1,\dots,l} \in \times \mathcal{S}^{\mathrm{P}}$$

2. Continue for next couple

Comments:

- Improves the convergence
- Low dimensional stochastic linear system $(l \times l)$
- Cost of update increases linearly with the order l of the reduced representation

Arnoldi, Full Update version

1. Set $l = 0$
2. Initialize $\lambda \in \mathcal{S}^{\mathrm{P}}$
3. For $l' = 1, 2, \ldots$ (Arnoldi iterations)
   - Solve deterministic problem $u' = \mathcal{D}(\lambda, U^l)$
   - Orthogonalize: $u_{l+l'} = u' - \sum_{j=1}^{l+l'-1} (u', u_j)_\Omega$
   - If $\|u_{l+l'}\|_{\mathrm{L}^2(\Omega)} \leq \epsilon$ or $l + l' = m$ then break
   - Normalize $u_{l+l'}$
   - Solve $\lambda = \mathcal{S}(u_{l'}, U^l)$
4. $l \leftarrow l + l'$
5. Find $\{\lambda_1, \ldots, \lambda_l\}$ s.t. (Update)

$$A\left(\sum_{i=1}^{l} u_i\lambda_i, \sum_{i=1}^{l} u_i\beta_i\right) = B\left(\sum_{i=1}^{l} u_i\beta_i\right), \quad \forall \beta_{i=1,\ldots,l} \in \mathcal{S}^{\mathrm{P}}$$

6. If $l < m$ return to step 2.

Summary

- Resolution of a sequence of deterministic elliptic problems, with elliptic coefficients $\mathbb{E}\left[\lambda^2 k\right]$ and modified (deflated) rhs

  dimension is $\dim \mathcal{V}^h$

- Resolution of a sequence of linear stochastic equations

  dimension is $\dim \mathcal{S}^{\mathrm{P}}$

- Update problems: system of linear equations for stochastic random variables

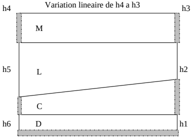
  dimension is $m \times \dim \mathcal{S}^{\mathrm{P}}$

- To be compared with the Galerkin problem dimension

  $\dim \mathcal{V}^h \times \dim \mathcal{S}^{\mathrm{P}}$

Weak modification of existing (FE/FV) codes
(weakly intrusive)

Test case definition : 25 x 0.695 km



Δ Head (m)	Expectation	Range	distribution
$\Delta h_{1,2}$	+51	± 10	Uniform
$\Delta h_{1,3}$	+21	± 5	Uniform
$\Delta h_{1,6}$	-3	± 2	Uniform
$\Delta h_{2,5}$	-110	± 10	Uniform
$\Delta h_{3,4}$	-160	± 20	Uniform

Uncertain conductivities

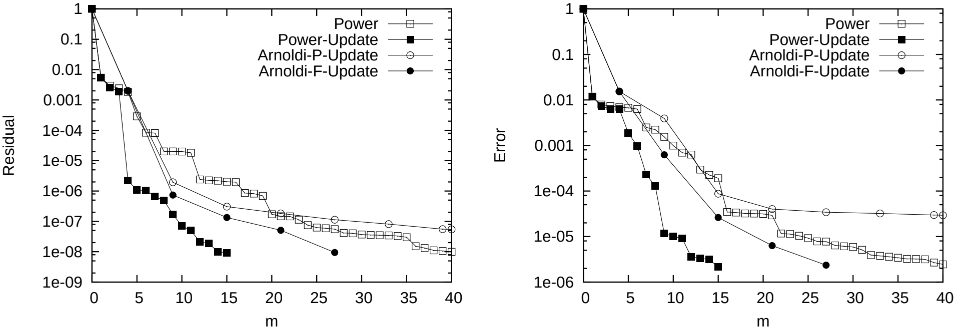
Layer	k_i median	k_i min	k_i max	distribution
Dogger	25	5	125	LogUniform
Clay	$3\ 10^{-6}$	$3\ 10^{-7}$	$3\ 10^{-5}$	LogUniform
Limestone	6	1.2	30	LogUniform
Marl	$3\ 10^{-5}$	$1\ 10^{-5}$	$1\ 10^{-4}$	LogUniform

Parameterization

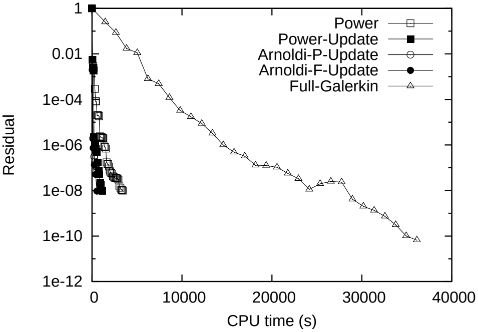
- 9 independent r.v. $\{\xi_1, \dots, \xi_9\} \sim U[0,1]^9$
- $\dim \mathcal{S}^{\mathrm{P}} = \mathrm{P} + 1 = (9 + \mathrm{No})!/(9!\mathrm{No}!)$
- $N_e \approx 30,000$ finite elements
- $\dim(\mathcal{V}^h) \approx 15,000$
- **Dimension of Galerkin problem:** $8.2\ 10^5$ $(\mathrm{No} = 2)$, $3.3\ 10^6$ $(\mathrm{No} = 3)$

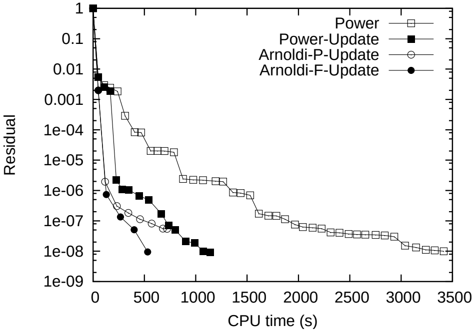
Convergence

Galerkin residual (left) and error (right) norms as a function of *m* ($\text{No} = 3$)



CPU times (No $= 3$)





Full separation

So far, deterministic / stochastic separation:

$$U^m(\boldsymbol{x},\boldsymbol{\xi}) = U^m(\boldsymbol{x},\xi_1,\dots,\xi_{\mathrm{N}}) = \sum_{r=1}^{m} u_r(\boldsymbol{x})\lambda_r(\xi_1,\dots,\xi_{\mathrm{N}}),$$

where $\lambda_r(\boldsymbol{\xi}) \in \mathcal{S}$.
Does not address high-dimensionality issue whenever N is large.

However, if the ξ_i are independent, $\mathcal{S}$ has a tensor product structure,

$$\mathcal{S} = \mathcal{S}_1 \otimes \cdots \otimes \mathcal{S}_{\mathrm{N}},$$

we can think of a decomposition of the form

$$U^m(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{r=1}^{m} u_r(\boldsymbol{x})\lambda_r^1(\xi_1)\dots\lambda_r^{\mathrm{N}}(\xi_{\mathrm{N}}),$$

where now $\lambda_r^i(\xi_i) \in \mathcal{S}_i$.

Full separation

Extension of the previous algorithms for the computation of

$$U^m(\boldsymbol{x},\boldsymbol{\xi}) = \sum_{r=1}^{m} u_r(\boldsymbol{x})\lambda_r^1(\xi_1)\dots\lambda_r^{\mathrm{N}}(\xi_{\mathrm{N}}),$$

is straightforward:

- same deterministic problems
- stochastic and update problems for the (separated) λ_r are substituted with **alternated direction resolutions**: iterations over sequence of one-dimensional problems.

For instance, stochastic problem(s) in direction i: find $\lambda \in \mathcal{S}_i$ such that

$$\mathbb{E}\left[\left(\lambda_r^1\dots\lambda\dots\lambda_r^{\mathrm{N}}\right)\left(\lambda_r^1\dots\beta\dots\lambda_r^{\mathrm{N}}\right)\int_\Omega k\boldsymbol{\nabla}u_r\cdot\boldsymbol{\nabla}u_r\mathrm{d}\boldsymbol{x}\right]$$
$$= \mathbb{E}\left[-\left(\lambda_r^1\dots\beta\dots\lambda_r^{\mathrm{N}}\right)\left(\int_\Omega k\boldsymbol{\nabla}U^{r-1}\cdot\boldsymbol{\nabla}u_r\mathrm{d}\boldsymbol{x} + \int_\Omega f u_r\mathrm{d}\boldsymbol{x}\right)\right], \quad \forall\beta\in\mathcal{S}_i.$$

Full separation

Clearly, using

$$U^m(\boldsymbol{x}, \xi) = \sum_{r=1}^{m} u_r(\boldsymbol{x}) \lambda_r^1(\xi_1) \dots \lambda_r^{\mathrm{N}}(\xi_{\mathrm{N}}),$$

we trade convergence with complexity reduction.

This can be mitigated using using a R_λ**-rank** approximation of the stochastic coefficients:

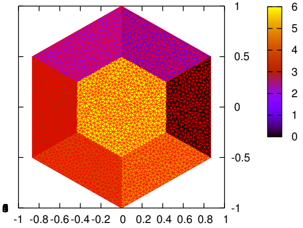
$$U^m(\boldsymbol{x}, \xi) = \sum_{r=1}^{m} u_r(\boldsymbol{x}) \left(\sum_{r'=1}^{R_\lambda} \lambda_{r,r'}^1(\xi_1) \dots \lambda_{r,r'}^{\mathrm{N}}(\xi_{\mathrm{N}}) \right),$$

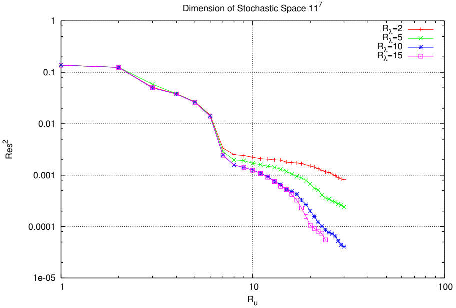
with a greedy-type approximation of low rank approximation of λ_r.

- Extension of the algorithms is immediate
- R_λ can be made rank dependent
- Efficient implementation requires separated representation of the operator.

An example: diffusion

- Independent random conductivities over 7 sub-domains, with same distribution (log-normal): $\mathrm{N} = 7$
- $\mathcal{S}_{i=1,7} = \Pi_{10}(\mathbb{R})$, so $\dim \mathcal{S} = 11^7$





Wave equation (Deterministic)

Consider the deterministic wave equation,

$$-\omega^2 \rho u(\boldsymbol{x}) - \boldsymbol{\nabla} \cdot (\tilde{\kappa} \boldsymbol{\nabla} u(\boldsymbol{x})) = f(\boldsymbol{x}), \qquad \text{in}\Omega$$
$$u(\boldsymbol{x} \in \partial\Omega) = 0$$

- ω is the frequency
- ρ the density
- $\tilde{\kappa} \doteq \kappa(1 - i\beta\omega) \in \mathbb{C}$ the wave velocity with $\kappa, \beta > 0$

Let $L_2(\Omega) = L_2(\Omega, \mathbb{C})$ with inner product and norm

$$(u, v)_\Omega = \mathrm{Re}\left(\int_\Omega u^*(\boldsymbol{x}) v(\boldsymbol{x}) d\Omega\right), \quad \|u\|^2_{L_2(\Omega)} = (u, u)_\Omega,$$

The weak formulation: Find $u \in H^1_0(\Omega, \mathbb{C})$ such that

$$a(u, v) - b(v) = 0 \quad \forall v \in H^1_0(\Omega),$$

with the bilinear and linear forms

$$a(u, v) = \mathrm{Re}\left[-\omega^2 \int_\Omega u^* v d\Omega + \int_\Omega \tilde{\kappa} \boldsymbol{\nabla} u^* \cdot \boldsymbol{\nabla} v \, d\Omega\right], \quad b(v) = \mathrm{Re}\left[\int_\Omega f^* v \, d\Omega\right].$$

Wave equation (Stochastic version)

Take now ω, ρ and κ as **second order random variable** defined on a probability space $\mathcal{P} = (\Theta, \Sigma_\Theta, \mu)$.
We extend $L_2(\Omega)$ and $H_0^1(\Omega)$ to $L_2(\Omega,\Theta)$ and $H_0^1(\Omega,\Theta)$ by tensorization, and we assume

$$U(\boldsymbol{x},\theta) \in L_2(\Omega,\Theta) \Leftrightarrow \mathbb{E}\left[(U(\cdot),U(\cdot))_\Omega\right] < \infty.$$

Variational form of the stochastic wave equation
Find $U \in H_0^1(\Omega,\Theta)$ such that

$$A(U,V) - B(V) = 0, \quad \forall V \in H_0^1(\Omega,\Theta),$$

where

$$A(U,V) = \mathbb{E}\left[\mathrm{Re}\left[-\omega^2(\theta)\int_\Omega U^*(\theta)V(\Theta)d\Omega + \int_\Omega \kappa(\theta)\boldsymbol{\nabla}U^*(\theta)\cdot\boldsymbol{\nabla}V(\theta)\,d\Omega\right]\right],$$

and

$$B(V) = \mathbb{E}\left[\mathrm{Re}\left[\int_\Omega f^* V(\theta)\,d\Omega\right]\right].$$

PGD approximation

We seek for $U \in H_0^1(\Omega, \Theta) = H_0^1(\Omega) \otimes L_2(\Theta)$ has the separated form

$$U(\boldsymbol{x}, \theta) = \sum_{r=0}^{r=\infty} u_r(\boldsymbol{x})\lambda_r(\theta), \quad u_r \in H_0^1(\Omega),\ \lambda_r \in L_2(\Theta),$$

following the PGD approach based on the deterministic and stochastic problems

$$\begin{aligned} u_R = D(U^{R-1}, \lambda_R): &\quad A(U^{R-1} + u_R\lambda_R, v\lambda_R) - B(v\lambda_R) = 0, \forall v \in H_0^1(\Omega) &\quad \text{Deter. problem} \\ \lambda_R = S(U^{R-1}, u_R): &\quad A(U^{R-1} + u_R\lambda_R, u_R\beta) - B(u_R\beta) = 0, \forall \beta \in L_2(\Theta) &\quad \text{Stoch. problem} \end{aligned}$$

and update problem:
given $u_{r=1,\ldots,R}$ compute $\lambda_{r=1,\ldots,R}$ such that

$$A\left(\sum_{r=0}^{R} u_r\lambda_r, u_{r'}\beta\right) - B(u_{r'}\beta) = 0, \quad \forall \beta \in L_2(\Theta) \text{ and } r' = 1, \ldots, R.$$

PGD-Arnoldi algorithm

Assume rank-R approximation has been obtained.

1. **Initialization:** set $\lambda \in L_2(\Theta)$, $l = 0$
2. **Arnoldi subspace generation:**
   - Set $w = D(U^R, \lambda)$
   - For $r = 1, \dots, R + l$ $w \leftarrow (w, u_r)_\Omega$
   - If $h = (w, w)_\Omega < \varepsilon$ break
   - Set $l \leftarrow l + 1$, $u_{R+l} = w/h$
   - Set $\lambda = S(U^R, u_{R+l})$
   - Repeat for next Arnoldi vector
3. **Update solution:** set $R \leftarrow R + l$ and solve

$$A\left(\sum_{r=0}^{R} u_r\lambda_r, u_{r'}\beta\right) - B(u_{r'}\beta) = 0, \quad \forall \beta \in L_2(\Theta) \text{ and } r' = 1, \dots, R.$$

4. **Check residual to restart at step 1 or stop**

Advantage: limited number of deterministic problem solves to generate the deterministic basis.

Stochastic parametrization

We introduce a **finite set of N independnt real-valued r.v. $\xi \doteq (\xi_1 \dots \xi_\mathrm{N})$** with **uniform distribution on $\Xi \doteq \mathbb{1}_N$**. The random frequency, density and stiffness are parametrized using $\boldsymbol{\xi}$,

$$(\omega, \kappa, \rho)(\theta) \longrightarrow (\omega, \kappa, \rho)(\boldsymbol{\xi}(\theta)),$$

and U is sought in the image probability space:

$$H_0^1(\Omega, \Xi) \ni U(\boldsymbol{x}, \boldsymbol{\xi}(\theta)) \approx \sum_{r=1}^{R} u_r(\boldsymbol{x}) \lambda_r(\boldsymbol{\xi}(\theta)).$$

- $U(\boldsymbol{x}, \dot{})$ is expected to be smooth a.s.: need for a limited number of spatial modes to span the stochastic solution space,
- $U(\cdot, \boldsymbol{\xi})$ can exhibit steep and complex dependences with respect to the input parameters.

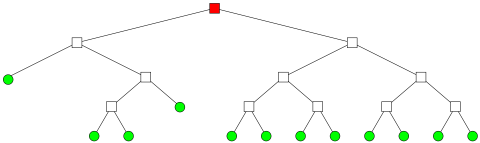
The complexity of the mapping $\boldsymbol{\xi} \in \Xi \mapsto U(\cdot, \boldsymbol{\xi}) \in H_0^1(\Omega)$ reflects in the stochastic coefficients $\lambda_r(\boldsymbol{\xi})$ and calls for **appropriate discretization at the stochastic level**.

stochastic multi-resolution framework

Presently, we use **piecewise polynomial approximations** at the stochastic level:

- Ξ is adaptively decomposed into sub-domains through a sequence a dyadic (1d) partitions
- A tree structure is used to manage the resulting stochastic space
- Multi-resolution analysis is used to control the local adaptation (anisotropic refinement of the partition of Ξ)
- Stochastic and update problems are solved independently over the sub-domains (efficient parallelization)

(see [Tryoen, LM and Ern, SISC 2012])



PGD-Arnoldi with Adaptation at the Stochastic level

Given the approximation U^r and a stochastic space $\mathcal{S}^r$

1. Arnoldi iterations to generate orthonormal $u_{r+1}, \ldots u_{r+l}$, using $\lambda \in \mathcal{S}^r$
2. set $r \leftarrow r + l$
3. While not satisfying accuracy criterion, repeat
   - Solve the update problem for $\{\lambda_1, \ldots, \lambda_r\}$ in $\mathcal{S}^r$
   - **Enrich adaptively $\mathcal{S}^r$**
4. Compute residual norm
5. If not converge restart at step 1.

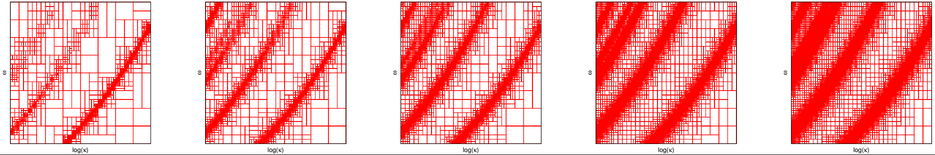
Observe:

- Same approximation space for all stochastic coefficients (ease implementation and favor parallelization)
- Continuous enrichment, no coarsening
- Successive Arnoldi spaces generated using an coarse stochastic space! (in fact robust)
- **Accuracy requirement should balance stochastic discretization and reduced space errors**.

Example

- $\log(\kappa) \sim U[-4:-2]$
- $\omega \sim U[0.5, 1]$
- $\rho = 1$ and $\beta = 0.05$
- Third order (Legendre) expansion.

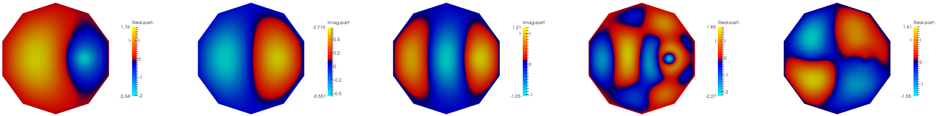
$r = 8$	$r = 13$	$r = 19$	$r = 26$	$r = 30$



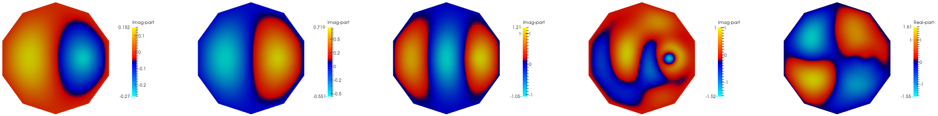
Example

Selected Arnoldi modes: real part (top) and imaginary part (bottom)

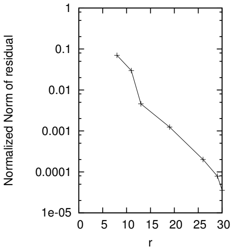
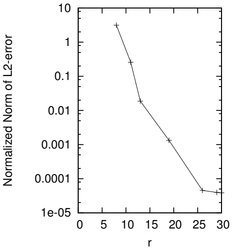
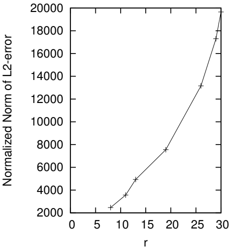
$r = 1$	$r = 3$	$r = 5$	$r = 15$	$r = 25$



$r = 1$	$r = 3$	$r = 5$	$r = 15$	$r = 25$



Example

Residual	Error	# of sub-domains
		

Stochastic Navier-Stokes equations

Consider the **steady, incompressible** Navier-Stokes equations

$$\begin{aligned}
&\boldsymbol{U}(\theta)\boldsymbol{\nabla}\boldsymbol{U}(\theta) = -\boldsymbol{\nabla}P(\theta) + \nu(\theta)\nabla^2\boldsymbol{U}(\theta) + \boldsymbol{f}(\theta) && \text{in } \Omega,\\
&\boldsymbol{\nabla}\cdot\boldsymbol{U}(\theta) = 0 && \text{in } \Omega,\\
&\boldsymbol{U}(\theta) = 0 && \text{on } \partial\Omega.
\end{aligned}$$

in a bounded (2d) domain Ω.
In view of PGD of the solution, we need to consider (mainly)

1. non-linear character (increases when $\nu \downarrow 0$)
2. enforcement of the divergence free constraint
3. stabilization (upwinding) due to the convective term

None of these will be really address here, simply numerical experiments!
`[Tamellini, LM, Nouy, SISC, 2014]`

Weak form

Deterministic space $\mathcal{V} = H^1_{0,div}(\Omega)$.

Weak formulation: Find $U \in \mathbb{X} \doteq \mathcal{V} \otimes \mathcal{S}$ such that

$$\mathbb{E}\left[\int_\Omega [(U(\theta)\nabla U(\theta)) \cdot V(\theta) + \nu(\theta)\nabla U(\theta)\ \nabla V(\theta) - F(\theta) \cdot V(\theta)]\, dx\right] \quad \forall V \in \mathbb{X}.$$

The deterministic problem $u = D(\lambda, U^m)$ writes: $\forall v \in \mathcal{V}$

$$\int_\Omega \left(\mathbb{E}\left[\lambda^3\right] u\nabla u + u\nabla \bar{u}_m(\lambda) + \bar{u}_m(\lambda)\nabla u\right) \cdot v dx + \int_\Omega \mathbb{E}\left[\nu\lambda^2\right] \nabla u \nabla v dx$$
$$= \int_\Omega \mathbb{E}\left[\lambda(F - U^m\nabla U^m)\right] \cdot v dx - \int_\Omega \mathbb{E}\left[\nu\lambda\nabla U^m\right]\nabla v dx.$$

where $\bar{u}_m(\lambda) = \mathbb{E}\left[\lambda^2 U^m\right]$.

Stochastic problem $\lambda = S(u, U^m)$ writes: $\forall \beta \in \mathcal{S}$

$$\mathbb{E}\left[\lambda^2\beta\right] \int_\Omega (u\nabla u \cdot u) dx + \mathbb{E}\left[\lambda\beta \int_\Omega (u\nabla U^m + U^m\nabla u) \cdot u dx\right] + \int_\Omega \mathbb{E}\left[\nu\lambda\beta\right]\nabla u \nabla u dx$$
$$= \mathbb{E}\left[\beta \int_\Omega (F - U^m\nabla U^m) \cdot u dx\right] - \mathbb{E}\left[\beta \int_\Omega \nu \nabla U^m \nabla u dx\right].$$

Complexity

- Resolution of a sequence of deterministic problems, NS + Lin. term and deflated rhs

  dimension is $\dim \mathcal{V}^h$

- Resolution of a sequence of quadratic stochastic equations

  dimension is $\dim \mathcal{S}$

- Update problems: system of quadratique equations for stochastic random variables

  dimension is $m \times \dim \mathcal{S}$

- To be compared with the Galerkin problem dimension

  $\dim \mathcal{V}^h \times \dim \mathcal{S}$

Weak modification of existing (FE/FV) codes
(weakly intrusive)

Stochastic discretization:

- **Parametrization** of $\nu(\theta)$ and $\boldsymbol{F}(\theta)$ using N i.i.d. random variables:

$$\boldsymbol{\xi} = \{\xi_1, \dots, \xi_{\mathrm{N}}\} \sim N(0, I^2).$$

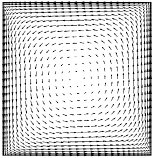
- Wiener-Hermite polynomials for the basis for $\mathcal{S}$

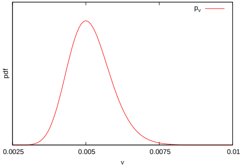
$$\lambda(\theta) = \sum_{\alpha} \lambda_\alpha \Psi_\alpha(\boldsymbol{\xi}(\theta)),$$

- **Truncature** to (total) polynomial degree No:

$$\dim \mathcal{S} = \frac{(\mathrm{No}+\mathrm{N})!}{\mathrm{No}!\mathrm{N}!}.$$

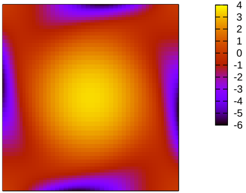
Case of a deterministic forcing and a random (Log-normal) viscosity:

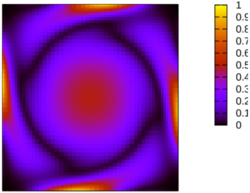




$$\nu(\theta) = \frac{1}{200} \exp\left(\frac{\sigma_\nu}{\sqrt{\mathrm{N}}} \sum_{i=1}^{\mathrm{N}} \xi_i(\theta) \right) \quad (+10^{-4}), \qquad \xi_i \sim N(0,1)\ i.i.d.$$

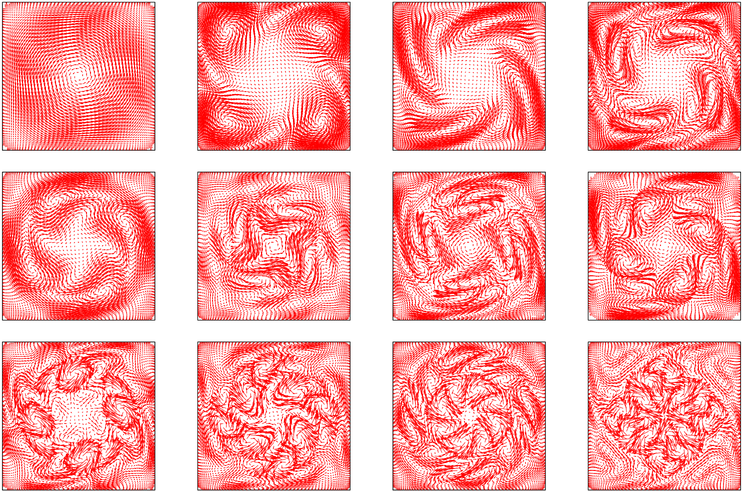
Same problem but for parametrization involving N Gaussian R.V.
Galerkin solution for $\mathrm{N} = 1$ and $\mathrm{No} = 10$ (Wiener-Hermite expansion)



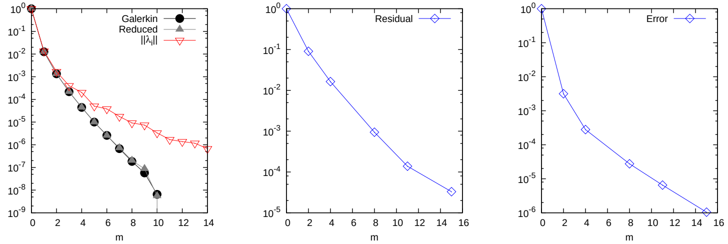


Mean and standard deviation of U^G rotational.

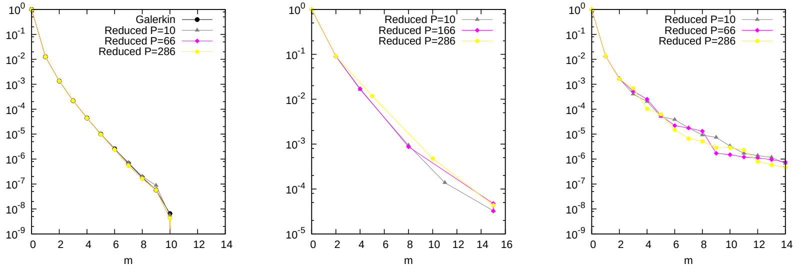
First PGD-Arnoldi modes for $\mathrm{N} = 1$ and $\mathrm{No} = 10$



Convergence of PGD solution $\mathrm{N} = 1$ and $\mathrm{No} = 10$



Convergence with rank of resiudal and error norms; POD coefficients at $m = 15$ (right)



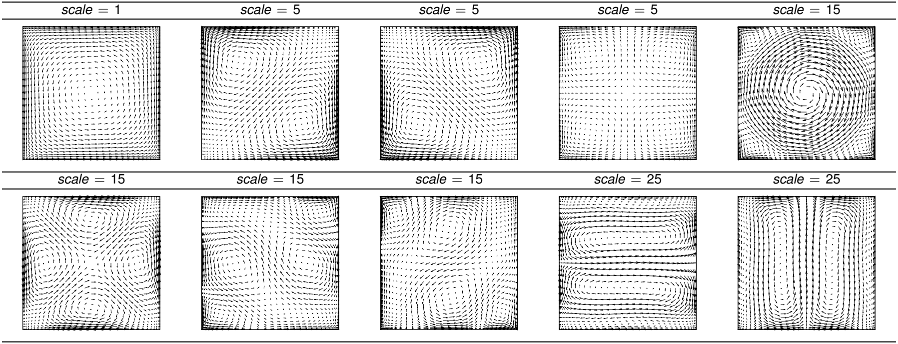
Norms of the POD coefficients at $m = 15$ (left), residual norm (center), $|\lambda|$'s norm (right).

PGD captures the essential features of the stochastic solution

Stochastic forcing $\boldsymbol{F}$: Hodge's decomposition

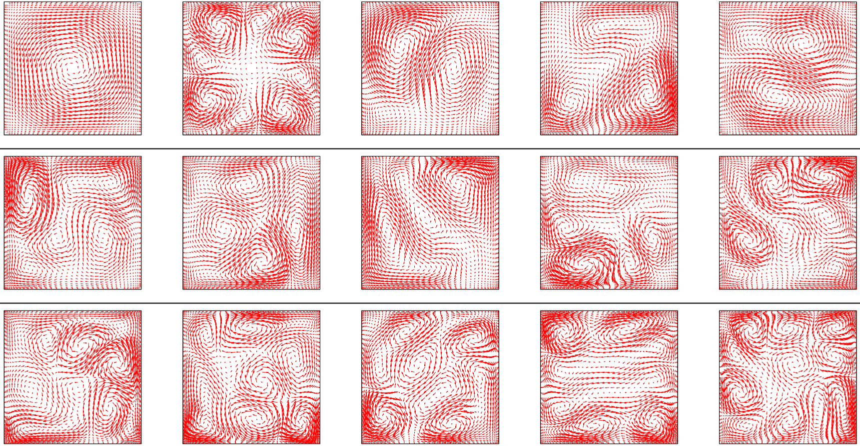
$$\boldsymbol{F}(\boldsymbol{x}, \theta) \approx \boldsymbol{F}^{\mathrm{N}}(\boldsymbol{x}, \boldsymbol{\xi}(\theta)) = \boldsymbol{f}^0 + \sum_{k=0}^{\mathrm{N}} \sqrt{\gamma_k} \boldsymbol{f}^k(\boldsymbol{x}) \xi_k(\theta).$$

KL modes of the forcing:

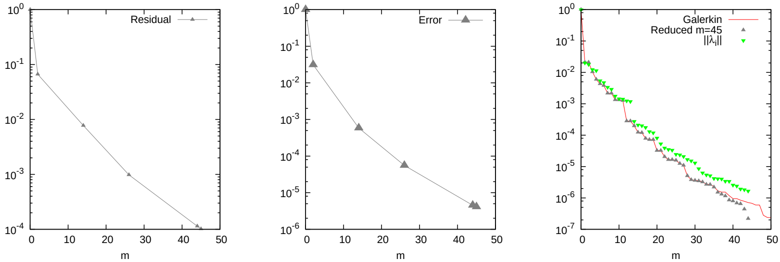


Forcing modes for $L = 1$, $\sigma / f_\omega^0 = 0.2$

First PGD-Arnoldi modes



Results at $\overline{\nu} = 1/50$: $\mathrm{No} = 3$, $\mathrm{N} = 11$, $\mathrm{P} = 364$

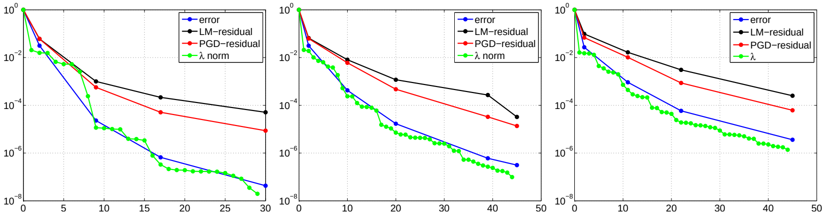


Residual (left), $\|U^m - U^G\|$ (center) and norm of POD modes for $m = 45$ (right).

Essentially < 50 Navier-Stokes solves!

Residual computation:

- computation of the residual in $H^1_{0,div}(\Omega)$
- need to reconstruct the pressure
- 2 alternatives: apply PGD to the pressure unknown, given the reduced velocity approximation, or recycle the pressure fields associated to the enforcement of the divergence-free constraint during the Arnoldi process as a reduced pressure basis.



Comparison of different error measures of the PGD solution at $\overline{\nu} = 1/10$, $1/50$ and $1/100$ (from left to right).

Questions & Discussion